Tutorial 1 Solutions

1). a) $tr(\hat{x}\hat{y}) = tr(\hat{y}\hat{x})$

$$I = \sum_i |a_i\rangle\langle a_i|$$

$\{|a_i\rangle\}$ complete Basis

$$tr(\hat{x}\hat{y}) = \sum_{a_i} \langle a_i|\hat{x}\,\hat{y}|a_i\rangle = \sum_{a_i a_j} \langle a_i|\hat{x}|a_j\rangle\langle a_j|\hat{y}|a_i\rangle$$

$$= \sum_{a_i a_j} \langle a_j|\hat{y}|\underbrace{a_i\rangle\langle a_i}_{= I}|\hat{x}|a_j\rangle$$

$$= \sum_{a_j} \langle a_j|\hat{y}\hat{x}|a_j\rangle = tr(\hat{y}\hat{x}).$$

c). $$\exp(i f(\hat{A})) = I + i f(\hat{A}) + \frac{1}{2!}(-1) f^2(\hat{A}) + \cdots$$

$\hat{A}|a_k\rangle = a_k|a_k\rangle$

$\{|a_k\rangle\} \rightarrow$ eigenbasis

$$= I + i\sum_k f(\hat{A})|a_k\rangle\langle a_k| + \frac{1}{2!} i^2 \sum_k f(\hat{A}) f(\hat{A}) |a_k\rangle\langle a_k| + \cdots$$

$$f(\hat{A})|a_k\rangle = \sum_l c_l \hat{A}^l |a_k\rangle = \sum_l c_l a_k^l |a_k\rangle$$

$$= f(a_k)|a_k\rangle \;//$$

f analytic function — power series

$$\therefore \exp(i f(\hat{A})) = \underbrace{I}_{=\sum_k |a_k\rangle\langle a_k|} + i\sum_k f(a_k)|a_k\rangle\langle a_k| + \frac{1}{2!} i^2 \sum_k f^2(a_k)|a_k\rangle\langle a_k|$$

$$= \sum_k |a_k\rangle\langle a_k| \left(1 + f(a_k) + \frac{i^2}{2!} f^2(a_k) + \cdots\right)$$

$$\boxed{e^{i f(\hat{A})} = \sum_k e^{i f(a_k)} |a_k\rangle\langle a_k|} \;//$$

2). a). $\|\psi - \alpha\phi\| = \sqrt{N(\alpha)}$

minimising N same as minimising $\|\psi - \alpha\phi\|$

$$N(\alpha) = \langle \psi - \alpha\phi | \psi - \alpha\phi \rangle$$

real LVS!

$$= \langle\psi|\psi\rangle - \alpha\langle\phi|\psi\rangle - \alpha\langle\psi|\phi\rangle + \alpha^2\langle\phi|\phi\rangle$$

$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle \in \mathbb{R}.$

$$\frac{dN}{d\alpha} = -2\langle\phi|\psi\rangle + 2\alpha\langle\phi|\phi\rangle = 0$$

$$\boxed{\alpha_m = \frac{\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}} \rightarrow \text{minima point.}$$

$$\frac{d^2N}{d\alpha^2} = 2\langle\phi|\phi\rangle \geq 0$$

$$\min_\alpha \|\psi - \alpha\phi\|^2 = \langle\psi|\psi\rangle - 2\frac{\langle\phi|\psi\rangle^2}{\langle\phi|\phi\rangle} + \frac{\langle\phi|\psi\rangle^2}{\langle\phi|\phi\rangle}$$

$$\min_\alpha \|\psi - \alpha\phi\| = \sqrt{\langle\psi|\psi\rangle - \frac{\langle\phi|\psi\rangle^2}{\langle\phi|\phi\rangle}} \ //.$$

b). $\|\psi - \alpha\phi\|^2 = N(\alpha, \alpha^*)$ $\qquad \alpha, \alpha^*$ lin. inde

$$\alpha = \alpha_1 + i\alpha_2$$

$$N(\alpha_1, \alpha_2) = \langle\psi - \alpha\phi | \psi - \alpha\phi\rangle$$

$$= \langle\psi|\psi\rangle - \alpha^*\langle\phi|\psi\rangle - \alpha\langle\psi|\phi\rangle + |\alpha|^2\langle\phi|\phi\rangle$$

$$= \langle\psi|\psi\rangle - \alpha_1(\langle\phi|\psi\rangle + \langle\psi|\phi\rangle) + i\alpha_2(\langle\phi|\psi\rangle - \langle\psi|\phi\rangle) + (\alpha_1^2 + \alpha_2^2)\langle\phi|\phi\rangle$$

$$\frac{\partial N}{\partial\alpha_1} = 0 \qquad -2\,\mathrm{Re}(\langle\phi|\psi\rangle) + 2\alpha_1\langle\phi|\phi\rangle = 0$$

$c + is$
$-c + is$

$$\alpha_{1m} = \frac{\mathrm{Re}(\langle\phi|\psi\rangle)}{\langle\phi|\phi\rangle} \qquad \alpha_{2m} = \frac{\mathrm{Im}(\langle\phi|\psi\rangle)}{\langle\phi|\phi\rangle}$$

$$\frac{\partial N}{\partial\alpha_2} = 0 \qquad -2\,\mathrm{Im}(\langle\phi|\psi\rangle) + 2\alpha_2\langle\phi|\phi\rangle = 0$$

$\therefore \quad \alpha_m = \alpha_{1m} + i\alpha_{2m} = \dfrac{\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}$

$$\frac{\partial^2 N}{\partial \alpha_1^2} = \frac{\partial^2 N}{\partial \alpha_2^2} = \langle\phi|\phi\rangle > 0$$

$$\det \begin{vmatrix} \partial^2 N/\partial\alpha_1^2 & \overset{\parallel\, 0}{\partial^2 N/\partial\alpha_1\partial\alpha_2} \\ \underset{\underset{0}{\parallel}}{\dfrac{\partial^2 N}{\partial\alpha_2\partial\alpha_1}} & \dfrac{\partial^2 N}{\partial\alpha_2^2} \end{vmatrix} > 0 \rightarrow \det \begin{vmatrix} \langle\phi|\phi\rangle & 0 \\ 0 & \langle\phi|\phi\rangle \end{vmatrix} > 0 \; ! \checkmark$$

$\therefore \quad \alpha_m = \dfrac{\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}$ location of minima → substitute in

$$\min_{\alpha\,\alpha^*} \|\psi - \alpha\phi\|^2 = \left[\|\psi\|^2\|\phi\|^2 - |\langle\phi|\psi\rangle|^2\right] / \|\phi\|^2$$

↓

Cauchy-Schwarz from

$\Rightarrow \|\psi - \alpha\phi\|^2 \geq 0$

(c) recall a in the original derivation is $-\alpha$ here,
no better bound possible, we have minimized $\|\psi + a\phi\|^2$!! //

3). a) $\hat{O} = |\alpha\rangle\langle\beta| = \sum_{a'a''} |a'\rangle\langle a'|\alpha\rangle\langle\beta|a''\rangle\langle a''|$

$$= \sum_{a'a''} \langle a'|\alpha\rangle\langle\beta|a''\rangle\, |a'\rangle\langle a''|$$

$$\therefore \quad \langle b'|\hat{O}|b''\rangle = \langle b'|\alpha\rangle\langle\beta|b''\rangle \; //.$$

b). $|\alpha\rangle = |S_z = \hbar/2\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$; $|\beta\rangle = |S_x = \hbar/2\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$|\alpha\rangle\langle\beta| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \; //.$$